

# Equilibrium in Two-Player Non-Zero-Sum Dynkin Games in Continuous Time\*

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## Abstract

We prove that every two-player non-zero-sum Dynkin game in continuous time admits an  $\varepsilon$ -equilibrium in randomized stopping times. We provide a condition that ensures the existence of an  $\varepsilon$ -equilibrium in non-randomized stopping times.

**Keywords:** Dynkin games, stopping games, equilibrium, stochastic analysis, continuous time.

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# 1 Introduction

Dynkin games (Dynkin, 1969) serve as a model of optimal stopping. These games were applied in various setups, including wars of attrition (see, e.g., Maynard Smith (1974), Ghemawat and Nalebuff (1985) and Hendricks et al. (1988)), pre-emption games (see, e.g., Fudenberg and Tirole (1991, section 4.5.3)), duels (see, e.g., Blackwell (1949), Bellman and Girshick (1949), Shapley (1951), Karlin (1959), and the survey by Radzik and Raghavan (1994)), and pricing of options (Kifer (2000), Hamadène (2006)).

The existence of a value in randomized strategies in Dynkin games, in its general setting, has been settled only recently (see Rosenberg, Solan and Vieille (2001) for discrete time games, and Laraki and Solan (2005), for continuous time games). The existence of an  $\varepsilon$ -equilibrium in randomized strategies in non-zero-sum games has been proven for two-player games in discrete time (Shmaya and Solan, 2004), and for games in continuous time under certain conditions (see, e.g., Laraki, Solan and Vieille, 2005).

In the present paper we prove that every two-player non-zero-sum Dynkin game in continuous time admits an  $\varepsilon$ -equilibrium in randomized strategies, for every  $\varepsilon > 0$ . We further show how such an equilibrium can be constructed, and we provide a condition under which there exists an  $\varepsilon$ -equilibrium in non-randomized strategies. Rather than using the Snell envelope, as, e.g., in Hamadène and Zhang (2010), our technique is to use results from zero-sum games.

We note that three-player Dynkin games in continuous time may fail to admit an  $\varepsilon$ -equilibrium in randomized strategies, even if the payoff processes are constant (Laraki, Solan and Vieille, 2005, Section 5.2). Thus, our result completes the mapping of Dynkin games in continuous time that admit an  $\varepsilon$ -equilibrium in randomized stopping times.

The paper is organized as follows. The model and the main results appear in Section 2. In Section 3 we review known results regarding zero-sum games that are then used in Section 4 to prove the main theorem.

## 2 Model and Results

Let  $(\Omega, \mathcal{A}, P)$  be a probability space, and let  $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$  be a filtration in continuous time that satisfies “the usual conditions”. That is,  $\mathcal{F}$  is right continuous, and  $\mathcal{F}_0$  contains all  $P$ -null sets: for every  $B \in \mathcal{A}$  with  $P(B) = 0$  and every  $A \subseteq B$ , one has  $A \in \mathcal{F}_0$ . All stopping times in the sequel are w.r.t. the filtration  $\mathcal{F}$ .

Denote  $\mathcal{F}_\infty := \vee_{t \geq 0} \mathcal{F}_t$ . Assume without loss of generality that  $\mathcal{F}_\infty = \mathcal{A}$ . Hence  $(\Omega, \mathcal{A}, P)$  is a complete probability space.

Let  $(X_i, Y_i, Z_i)_{i=1,2}$  be uniformly bounded  $\mathcal{F}$ -adapted real-valued processes,<sup>1</sup> and let  $(\xi_i)_{i=1,2}$  be two bounded real-valued  $\mathcal{F}_\infty$ -measurable functions. In the sequel we will assume that the processes  $(X_i, Y_i, Z_i)_{i=1,2}$  are right continuous.

**Definition 1** *A two-player non-zero-sum Dynkin game over  $(\Omega, \mathcal{A}, P, \mathcal{F})$  with payoffs  $(X_i, Y_i, Z_i, \xi_i)_{i=1,2}$  is the game with player set  $N = \{1, 2\}$ , the set of pure strategies of each player is the set of stopping times, and the payoff function of each player  $i \in \{1, 2\}$  is:*

$$\gamma_i(\lambda_1, \lambda_2) := \mathbf{E} \left[ X_i(\lambda_1) \mathbf{1}_{\{\lambda_1 < \lambda_2\}} + Y_i(\lambda_2) \mathbf{1}_{\{\lambda_2 < \lambda_1\}} + Z_i(\lambda_1) \mathbf{1}_{\{\lambda_1 = \lambda_2 < \infty\}} + \xi_i \mathbf{1}_{\{\lambda_1 = \lambda_2 = \infty\}} \right], \quad (1)$$

where  $\lambda_1$  and  $\lambda_2$  are the stopping times chosen by the two players respectively.

In words, the process  $X_i$  represents the payoff to player  $i$  if player 1 stops before player 2, the process  $Y_i$  represents the payoff to player  $i$  if player 2 stops before player 1, the process  $Z_i$  represents the payoff to player  $i$  if the two players stop simultaneously, and the function  $\xi_i$  represents the payoff to player  $i$  if no player ever stops.

The game is *zero-sum* if  $X_1 + X_2 = Y_1 + Y_2 = Z_1 + Z_2 = \xi_1 + \xi_2 = 0$ .

In non-cooperative game theory, a randomized strategy is a probability distribution over pure strategies, with the interpretation that at the outset of the game the player randomly chooses a pure strategy according to the probability distribution given by the randomized strategy, and uses it along the game. In the setup of Dynkin games in continuous time, a randomized strategy is a randomized stopping time, which is defined as follows.

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<sup>1</sup>Our results hold for the larger class of  $\mathcal{D}$  payoff processes defined by Dellacherie and Meyer, 1975, §II-18. This class contains in particular integrable processes.

**Definition 2** A randomized stopping time for player  $i$  is a measurable function  $\varphi_i : [0, 1] \times \Omega \rightarrow [0, +\infty]$  such that the function  $\varphi_i(r, \cdot) : \Omega \rightarrow [0, +\infty]$  is a stopping time for every  $r \in [0, 1]$  (see Aumann (1964)).

Here the interval  $[0, 1]$  is endowed with the Borel  $\sigma$ -field. For strategically equivalent definitions of randomized stopping times, see Touzi and Vieille (2002). The interpretation of Definition 2 is that player  $i$  chooses  $r$  in  $[0, 1]$  according to the uniform distribution and then stops at the stopping time  $\varphi_i(r, \cdot)$ . Throughout the paper, the symbols  $\lambda$ ,  $\mu$  and  $\tau$  stand for stopping times, and  $\varphi$  and  $\psi$  stand for randomized stopping times.

The *expected payoff* for player  $i$  that corresponds to a pair of randomized stopping times  $(\varphi_1, \varphi_2)$  is:

$$\gamma_i(\varphi_1, \varphi_2) := \int_{[0,1]^2} \gamma_i(\varphi_1(r, \cdot), \varphi_2(s, \cdot)) \, dr \, ds, \quad i = 1, 2.$$

In the sequel we will also consider the expected payoff at a given time  $t$ . We therefore define for every  $t \geq 0$  and every pair of randomized stopping times  $\varphi_1, \varphi_2 \geq t$ :

$$\gamma_i(\varphi_1, \varphi_2 \mid \mathcal{F}_t) := \mathbf{E}[X_i(\varphi_1)\mathbf{1}_{\{\varphi_1 < \varphi_2\}} + Y_i(\varphi_2)\mathbf{1}_{\{\varphi_2 < \varphi_1\}} + Z_i(\varphi_1)\mathbf{1}_{\{\varphi_1 = \varphi_2 < \infty\}} + \xi_i\mathbf{1}_{\{\varphi_1 = \varphi_2 = \infty\}} \mid \mathcal{F}_t]. \quad (2)$$

A pair of randomized stopping times  $(\varphi_1^*, \varphi_2^*)$  is an  $\varepsilon$ -equilibrium if no player can profit more than  $\varepsilon$  by deviating from  $\varphi_i^*$ .

**Definition 3** Let  $\varepsilon \geq 0$ . A pair of randomized stopping times  $(\varphi_1^*, \varphi_2^*)$  is an  $\varepsilon$ -equilibrium if for every two randomized stopping times  $\varphi_1, \varphi_2$  the following inequalities hold:

$$\gamma_1(\varphi_1, \varphi_2^*) \leq \gamma_1(\varphi_1^*, \varphi_2^*) + \varepsilon, \quad (3)$$

and

$$\gamma_2(\varphi_1^*, \varphi_2) \leq \gamma_2(\varphi_1^*, \varphi_2^*) + \varepsilon. \quad (4)$$

Because of the linearity of the payoff function, Eqs. (3) and (4) hold for every randomized stopping time  $\varphi_1$  and  $\varphi_2$  respectively as soon as they hold for non-randomized stopping times.

Our goal in this paper is to prove the existence of an  $\varepsilon$ -equilibrium in two-player non-zero-sum games, and to construct such an  $\varepsilon$ -equilibrium.

Suppose that a player wants to stop at the stopping time  $\lambda$ , but he would like to mask the exact time at which he stops (for example, so that the other player cannot stop at the very same moment as he does). To this end, he can stop at a randomly chosen time in a small interval  $[\lambda, \lambda + \delta]$ , and, since the payoff processes are right continuous, he will not lose (or gain) much relative to stopping at time  $\lambda$ . This leads us to the following class of simple randomized stopping times that will be extensively used in the sequel.

**Definition 4** *A randomized stopping time  $\varphi$  is simple if there exist a stopping time  $\lambda$  and a  $\mathcal{F}_\lambda$ -measurable non-negative function  $\delta \geq 0$ , such that for every  $r \in [0, 1]$  one has  $\varphi(r, \cdot) = \lambda + r\delta$ . The stopping time  $\lambda$  is called the basis of  $\varphi$ , and the function  $\delta$  is called the delay of  $\varphi$ .*

Since  $\varphi(r, \cdot) \geq \lambda$  and  $\varphi(r, \cdot)$  is  $\mathcal{F}_\lambda$ -measurable, by Dellacherie and Meyer (1975, §IV-56),  $\varphi(r, \cdot)$  is a stopping time for every  $r \in [0, 1]$ . Consequently,  $\varphi$  is indeed a randomized stopping time.

Definition 4 does not require that  $\lambda$  is finite:<sup>2</sup> on the set  $\{\lambda = \infty\}$  we have  $\varphi(r, \cdot) = \infty$  for every  $r \in [0, 1]$ . On the set  $\{\delta = 0\}$  the randomized stopping time  $\varphi$  that is defined in Definition 4 stops at time  $\lambda$  with probability 1. On the set  $\{\delta > 0\}$  the stopping time is “non-atomic” yet finite, and in particular for every stopping time  $\mu$  we have  $\mathbf{P}(\{\delta > 0\} \cap \{\varphi = \mu\}) = 0$ .

We now state our main results.

**Theorem 5** *Every two-player non-zero-sum Dynkin game with right-continuous and uniformly bounded payoff processes admits an  $\varepsilon$ -equilibrium in simple randomized stopping times, for every  $\varepsilon > 0$ .*

Moreover, the delay of the simple randomized stopping time that constitute the  $\varepsilon$ -equilibrium can be arbitrarily small.

Theorem 5 was proved by Laraki and Solan (2005) for two-player *zero-sum* games. Our proof heavily relies on the results of Laraki and Solan (2005), and we use  $\varepsilon$ -equilibria in zero-sum games to construct an  $\varepsilon$ -equilibrium in the non-zero-sum game.

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<sup>2</sup>A statement *holds* on a measurable set  $A$  if and only if the set of points in  $A$  that do not satisfy the statement has probability 0.

Under additional conditions on the payoff processes, the  $\varepsilon$ -equilibrium is given in non-randomized stopping times.

**Theorem 6** *Under the assumptions of Theorem 5, if  $Z_1(t) \in \text{co}\{X_1(t), Y_1(t)\}$  and  $Z_2(t) \in \text{co}\{X_2(t), Y_2(t)\}$  for every  $t \geq 0$ , then the game admits an  $\varepsilon$ -equilibrium in non-randomized stopping times, for every  $\varepsilon > 0$ .*

Hamadène and Zhang (2010) proved the existence of a 0-equilibrium in non-randomized stopping times under stronger conditions than those in Theorem 6, using the notion of Snell envelope of processes (see, e.g., El-Karoui (1980) for more details).

The rest of the paper is devoted to the proofs of Theorems 5 and 6. We will assume w.l.o.g. that the payoff processes are bounded between 0 and 1.

### 3 The Zero-Sum Case

In the present section we summarize several results on zero-sum games, taken from Laraki and Solan (2005), that will be used in the sequel, and prove some new results on zero-sum games.

For every  $t \geq 0$  denote

$$\begin{aligned} v_1(t) &:= \text{ess} - \sup_{\varphi_1 \geq t} \text{ess} - \inf_{\lambda_2 \geq t} \mathbf{E}[X_1(\varphi_1) \mathbf{1}_{\{\varphi_1 < \lambda_2\}} + Y_1(\lambda_2) \mathbf{1}_{\{\lambda_2 < \varphi_1\}} \\ &\quad + Z_1(\varphi_1) \mathbf{1}_{\{\varphi_1 = \lambda_2 < \infty\}} + \xi_1 \mathbf{1}_{\{\varphi_1 = \lambda_2 = \infty\}} \mid \mathcal{F}_t], \end{aligned} \quad (5)$$

where the supremum is over all randomized stopping times  $\varphi_1 \geq t$ , and the infimum is over all (non-randomized) stopping times  $\lambda_2 \geq t$ . This is the highest payoff that player 1 can guarantee in the zero-sum Dynkin game  $\Gamma_1(t)$  where the payoffs are those of player 1, player 1 is the maximizer, player 2 is the minimizer, and the game starts at time  $t$ . Similarly, the highest payoff that player 2 can guarantee in the zero-sum Dynkin game  $\Gamma_2(t)$  where the payoffs are those of player 2, player 2 is the maximizer, player 1 is the minimizer, and the game starts at time  $t$ , is given by:

$$\begin{aligned} v_2(t) &:= \text{ess} - \sup_{\varphi_2 \geq t} \text{ess} - \inf_{\lambda_1 \geq t} \mathbf{E}[X_2(\lambda_1) \mathbf{1}_{\{\lambda_1 < \varphi_2\}} + Y_2(\varphi_2) \mathbf{1}_{\{\varphi_2 < \lambda_1\}} \\ &\quad + Z_2(\lambda_1) \mathbf{1}_{\{\lambda_1 = \varphi_2 < \infty\}} + \xi_2 \mathbf{1}_{\{\lambda_1 = \varphi_2 = \infty\}} \mid \mathcal{F}_t]. \end{aligned} \quad (6)$$

The next lemma, which is proved in Laraki and Solan (2005), states that  $v_1(t)$  (resp.  $v_2(t)$ ) is in fact the value of the zero-sum games  $\Gamma_1(t)$  (resp.  $\Gamma_2(t)$ ). This lemma is proved in Laraki and Solan (2005) when  $\mathcal{F}_t$  is the trivial  $\sigma$ -algebra. Its proof can be adapted to a general  $\mathcal{F}_t$  (see the discussion in Appendix A).

**Lemma 7**

$$\begin{aligned} v_1(t) = & \text{ess} - \inf_{\psi_2 \geq t} \text{ess} - \sup_{\lambda_1 \geq t} \mathbf{E}[X_1(\lambda_1) \mathbf{1}_{\{\lambda_1 < \psi_2\}} + Y_1(\psi_2) \mathbf{1}_{\{\psi_2 < \lambda_1\}} \\ & + Z_1(\lambda_1) \mathbf{1}_{\{\lambda_1 = \psi_2 < \infty\}} + \xi_1 \mathbf{1}_{\{\lambda_1 = \psi_2 = \infty\}} \mid \mathcal{F}_t], \end{aligned} \quad (7)$$

and

$$\begin{aligned} v_2(t) = & \text{ess} - \inf_{\psi_1 \geq t} \text{ess} - \sup_{\lambda_2 \geq t} \mathbf{E}[X_2(\psi_1) \mathbf{1}_{\{\psi_1 < \lambda_2\}} + Y_2(\lambda_2) \mathbf{1}_{\{\lambda_2 < \psi_1\}} \\ & + Z_2(\psi_1) \mathbf{1}_{\{\psi_1 = \lambda_2 < \infty\}} + \xi_2 \mathbf{1}_{\{\psi_1 = \lambda_2 = \infty\}} \mid \mathcal{F}_t], \end{aligned} \quad (8)$$

where the infimum in (7) is over all randomized stopping times  $\psi_2 \geq t$  for player 2, the supremum in (7) is over all (non-randomized) stopping times  $\lambda_1 \geq t$  for player 1, the infimum in (8) is over all randomized stopping times  $\psi_1 \geq t$  for player 1, and the supremum in (8) is over all (non-randomized) stopping times  $\lambda_2 \geq t$  for player 2.

A stopping time  $\varphi_1$  (resp.  $\psi_1$ ) that achieves the supremum in (5) (resp. infimum in (8)) up to  $\varepsilon$  is called an  $\varepsilon$ -optimal stopping time for player 1 in  $\Gamma_1(t)$  (resp.  $\Gamma_2(t)$ ). Similarly, a stopping time  $\varphi_2$  (resp.  $\psi_2$ ) that achieves the supremum in (6) (resp. infimum in (7)) up to  $\varepsilon$  is called an  $\varepsilon$ -optimal stopping time for player 2 in  $\Gamma_2(t)$  (resp.  $\Gamma_1(t)$ ).

The proof of Laraki and Solan (2005, Proposition 7) can be adapted to show that the value process is right continuous (see Appendix A).

**Lemma 8** *The process  $(v_i(t))_{t \geq 0}$  is right continuous, for each  $i \in \{1, 2\}$ .*

The following two lemmas provide crude bounds on the value process.

**Lemma 9** *For every  $t \geq 0$  and each  $i = 1, 2$  one has*

$$\min\{X_i(t), Y_i(t)\} \leq v_i(t) \leq \max\{X_i(t), Y_i(t)\} \text{ on } \Omega.$$

**Proof.** We start by proving the left-hand side inequality for  $i = 2$ . Let  $\varepsilon > 0$  be arbitrary, and let  $\delta > 0$  be sufficiently small such that

$$\mathbf{P}(\sup_{\rho \in [0, \delta]} |X_2(t) - X_2(t + \rho)| > \varepsilon) \leq \varepsilon, \quad (9)$$

$$\mathbf{P}(\sup_{\rho \in [0, \delta]} |Y_2(t) - Y_2(t + \rho)| > \varepsilon) \leq \varepsilon. \quad (10)$$

Such  $\delta$  exists because the processes  $X_2$  and  $Y_2$  are right continuous.

Let  $\varphi_2$  be the simple randomized stopping time  $\varphi_2(r, \cdot) = t + r\delta$ , and let  $\lambda_1 \geq t$  be any non-randomized stopping time for player 1. The definition of  $\varphi_2$  implies that the probability that  $\lambda_1 = \varphi_2$  is 0:  $\mathbf{P}(\lambda_1 = \varphi_2) = 0$ . Moreover,  $\varphi_2 < \infty$ . Therefore

$$\gamma_2(\lambda_1, \varphi_2 \mid \mathcal{F}_t) = \mathbf{E}[X_2(\lambda_1)\mathbf{1}_{\{\lambda_1 < \varphi_2\}} + Y_2(\varphi_2)\mathbf{1}_{\{\varphi_2 < \lambda_1\}} \mid \mathcal{F}_t].$$

By (9) and (10) this implies that

$$\mathbf{P}(\gamma_2(\lambda_1, \varphi_2 \mid \mathcal{F}_t) < \min\{X_2(t), Y_2(t)\} - \varepsilon) \leq 2\varepsilon.$$

Because  $\lambda_1$  is arbitrary, Eq. (6) implies that

$$\mathbf{P}(v_2(t) < \min\{X_2(t), Y_2(t)\} - \varepsilon) \leq 2\varepsilon.$$

The left-hand side inequality for  $i = 2$  follows because  $\varepsilon$  is arbitrary.

The proof of the right-hand side-inequality for  $i = 2$  follows the same arguments, by using the simple randomized stopping time  $\varphi_1(r, \cdot) = t + r\delta$ . Indeed, for every stopping time  $\lambda_2$  for player 2 we then have

$$\gamma_2(\varphi_1, \lambda_2 \mid \mathcal{F}_t) = \mathbf{E}[X_2(\varphi_1)\mathbf{1}_{\{\varphi_1 < \lambda_2\}} + Y_2(\lambda_2)\mathbf{1}_{\{\varphi_1 > \lambda_2\}} \mid \mathcal{F}_t].$$

The same argument as above, using (8), delivers the desired inequality. The proof for  $i = 1$  is analogous. ■

**Lemma 10** *For every  $t \geq 0$ , one has*

$$\begin{aligned} v_1(t) &\leq \max\{Y_1(t), Z_1(t)\} \text{ on } \Omega, \\ v_2(t) &\leq \max\{X_2(t), Z_2(t)\} \text{ on } \Omega. \end{aligned}$$



**Proof.** We prove the Lemma for  $i = 1$ . Let  $\psi_2 = t$ : player 2 stops at time  $t$ . By (7),

$$v_1(t) \leq \text{ess} - \sup_{\lambda_1 \geq t} \gamma_1(\lambda_1, \psi_2 \mid \mathcal{F}_t).$$

Because for every (non-randomized) stopping time  $\lambda_1$  for player 1,  $\gamma_1(\lambda_1, \psi_2 \mid \mathcal{F}_t)$  is either  $Y_1(t)$  (if  $\lambda_1 > t$ ) or  $Z_1(t)$  (if  $\lambda_1 = t$ ), the result follows. ■

Following Lepeltier and Maingueneau (1984), for every  $\eta > 0$  let  $\mu_1^\eta$  and  $\mu_2^\eta$  be the stopping times defined as follows:

$$\mu_1^\eta := \inf\{s \geq 0 : X_1(s) \geq v_1(s) - \eta\}, \quad (11)$$

and

$$\mu_2^\eta := \inf\{s \geq 0 : Y_2(s) \geq v_2(s) - \eta\}. \quad (12)$$

As the following example shows, the stopping times  $\mu_1^\eta$  and  $\mu_2^\eta$  may be infinite. Consider the following Dynkin game, where the payoffs are constants:  $X_1 = 0$ ,  $Y_1 = Z_1 = 2$  and  $\xi_1 = 1$ . Then  $v_1(t) = 1$  for every  $t$ , and  $\mu_1^\eta = \infty$ , provided  $\eta \in (0, 1)$ .

Observe that  $\mu_2^\eta \leq \mu_2^{\eta'}$  whenever  $\eta > \eta'$ . Moreover, because the processes  $X_1$ ,  $Y_2$ ,  $v_1$  and  $v_2$  are right continuous, we have

$$X_1(\mu_1^\eta) \geq v_1(\mu_1^\eta) - \eta, \quad (13)$$

and

$$Y_2(\mu_2^\eta) \geq v_2(\mu_2^\eta) - \eta. \quad (14)$$

For every  $t < \mu_1^\eta$ , by the definition of  $\mu_1^\eta$  and Lemma 9, we have

$$X_1(t) < v_1(t) - \eta < v_1(t) \leq \max\{X_1(t), Y_1(t)\},$$

and therefore

$$Y_1(t) > X_1(t), \quad \forall t < \mu_1^\eta. \quad (15)$$

The analogous inequality for player 2 holds as well.

**Lemma 11** *Let  $\varepsilon, \eta > 0$ , let  $\tau$  be a stopping time, and let  $A \in \mathcal{F}_\tau$  satisfy  $\mathbf{P}(A \setminus \{\mu_1^\eta = \infty\}) < \varepsilon$ . Then*

$$\mathbf{E}[v_1(\tau)\mathbf{1}_A] \leq \mathbf{E}[\xi_1\mathbf{1}_{A \cap \{\mu_1^\eta = \infty\}}] + 3\varepsilon + 6\varepsilon/\eta. \quad (16)$$

**Proof.** Let  $\psi_2 = \infty$ : player 2 never stops. By (7),

$$v_1(\tau) \leq \text{ess} - \sup_{\lambda_1 \geq \tau} \gamma_1(\lambda_1, \psi_2 \mid \mathcal{F}_\tau). \quad (17)$$

Let  $\lambda_1 \geq \tau$  be a stopping time for player 1 that achieves the supremum in (17) up to  $\varepsilon$ . Let  $\lambda'_1$  be the following stopping time:

- On  $A \cap \{\lambda_1 < \infty\}$ ,  $\lambda'_1$  is an  $\eta/2$ -optimal stopping time for player 1 in  $\Gamma_1(\lambda_1)$ .
- On  $A \cap \{\lambda_1 = \infty\}$ ,  $\lambda'_1 = \infty$ .

It follows that

$$\begin{aligned} \mathbf{E}[v_1(\tau)\mathbf{1}_A] &\leq \mathbf{E}[\gamma_1(\lambda_1, \psi_2 \mid \mathcal{F}_\tau)\mathbf{1}_A] + \varepsilon \\ &= \mathbf{E}[X_1(\lambda_1)\mathbf{1}_{A \cap \{\lambda_1 < \infty\}} + \xi_1\mathbf{1}_{A \cap \{\lambda_1 = \infty\}}] + \varepsilon \\ &< \mathbf{E}[(v_1(\lambda_1) - \eta)\mathbf{1}_{A \cap \{\lambda_1 < \mu_1^\eta = \infty\}} + X_1(\lambda_1)\mathbf{1}_{A \cap \{\lambda_1 < \infty\} \cap \{\mu_1^\eta < \infty\}} + \xi_1\mathbf{1}_{A \cap \{\lambda_1 = \infty\}}] + \varepsilon \\ &\leq \mathbf{E}[(v_1(\lambda_1) - \eta)\mathbf{1}_{A \cap \{\lambda_1 < \infty\}} + \xi_1\mathbf{1}_{A \cap \{\lambda_1 = \infty\}}] + 3\varepsilon \\ &\leq \mathbf{E}[\gamma_1(\lambda'_1, \psi_2 \mid \mathcal{F}_\tau)\mathbf{1}_A] - \frac{\eta}{2}\mathbf{E}[\mathbf{1}_{A \cap \{\lambda_1 < \infty\}}] + 3\varepsilon \\ &\leq \mathbf{E}[\gamma_1(\lambda_1, \psi_2 \mid \mathcal{F}_\tau)\mathbf{1}_A] - \frac{\eta}{2}\mathbf{E}[\mathbf{1}_{A \cap \{\lambda_1 < \infty\}}] + 4\varepsilon \end{aligned}$$

where the second inequality holds by the definition of  $\mu_1^\eta$ , the third inequality holds since  $\mathbf{P}(A \setminus \{\mu_1^\eta = \infty\}) < \varepsilon$  and since payoffs are bounded by 1, and the last inequality holds because  $\lambda_1$  is  $\varepsilon$ -optimal.

This sequence of inequalities implies that

$$\mathbf{P}(A \cap \{\lambda_1 < \infty\}) \leq 6\varepsilon/\eta,$$

and therefore

$$\mathbf{E}[v_1(\tau)\mathbf{1}_A] \leq \mathbf{E}[\xi_1\mathbf{1}_{A \cap \{\mu_1^\eta = \infty\}}] + 3\varepsilon + 6\varepsilon/\eta,$$

as desired. ■

By Lepeltier and Maingueneau (1984), for each  $i = 1, 2$  the process  $v_i$  is a submartingale up to time  $\mu_i^\eta$ .

**Lemma 12** *For every  $\eta > 0$  the process  $(v_1(t))_{t=0}^{\mu_1^\eta}$  is a submartingale: for every pair of finite stopping times  $\lambda < \lambda' \leq \mu_1^\eta$  one has  $v_1(\lambda) \leq \mathbf{E}[v_1(\lambda') \mid \mathcal{F}_\lambda]$  on  $\Omega$ .*

Lemma 12 implies that before time  $\sup_{\eta>0} \mu_1^\eta$  player 1 is better off waiting and not stopping. An analogue statement holds for player 2.

Lemmas 11 and 12 deliver the following result.

**Lemma 13** *Let  $\eta > 0$ . For every stopping time  $\lambda_1$  that satisfies  $\lambda_1 \leq \mu_1^\eta$  one has*

$$v_1(\lambda_1) \leq \mathbf{E}[v_1(\mu_1^\eta) \mathbf{1}_{\{\mu_1^\eta < \infty\}} + \xi_1 \mathbf{1}_{\{\mu_1^\eta = \infty\}} \mid \mathcal{F}_{\lambda_1}].$$

**Proof.** Let  $\varepsilon > 0$  be arbitrary. By Lemma 12, for every  $t \geq 0$  one has

$$v_1(\lambda_1) \leq \mathbf{E}[v_1(\min\{\mu_1^\eta, t\})].$$

Let  $t_0$  be sufficiently large such that  $\mathbf{P}(t_0 \leq \mu_1^\eta < \infty) < \varepsilon$ . By Lemma 11 with  $\tau = t_0$  and  $A = \{t_0 \leq \mu_1^\eta\}$ ,

$$\mathbf{E}[v_1(t_0) \mathbf{1}_{\{t_0 \leq \mu_1^\eta\}}] \leq \mathbf{E}[\xi_1 \mathbf{1}_{\{t_0 \leq \mu_1^\eta = \infty\}}] + 3\varepsilon + 6\varepsilon/\eta.$$

Therefore,

$$\begin{aligned} v_1(\lambda_1) &\leq \mathbf{E}[v_1(\min\{\mu_1^\eta, t_0\})] \\ &= \mathbf{E}[v_1(\mu_1^\eta) \mathbf{1}_{\{\mu_1^\eta < t_0\}} + v_1(t_0) \mathbf{1}_{\{t_0 \leq \mu_1^\eta\}}] \\ &\leq \mathbf{E}[v_1(\mu_1^\eta) \mathbf{1}_{\{\mu_1^\eta < \infty\}} + \xi_1 \mathbf{1}_{\{\mu_1^\eta = \infty\}}] + 5\varepsilon + 6\varepsilon/\eta. \end{aligned}$$

The result follows since  $\varepsilon$  is arbitrary. ■

The proof of Laraki and Solan (2005, Section 3.3) delivers the following result, which says that each player  $i$  has a simple randomized  $\varepsilon$ -optimal stopping time that is based on  $\mu_i^\eta$ , provided  $\eta$  is sufficiently small.

**Lemma 14** *For every  $i = 1, 2$ , every  $\varepsilon, \eta > 0$ , and every positive  $\mathcal{F}_{\mu_i^\eta}$ -measurable function  $\delta_i$ , there exists a simple randomized stopping time  $\varphi_i^\eta$  with basis  $\mu_i^\eta$  and delay at most  $\delta_i$  that satisfies*

$$\gamma_i(\varphi_i^\eta, \lambda_{3-i} \mid \mathcal{F}_{\mu_i^\eta}) \geq v_i(\mu_i^\eta) - \varepsilon - \eta \text{ on } \Omega, \quad (18)$$

*for every stopping time  $\lambda_{3-i} \geq \mu_i^\eta$ .*

By Eq. (15), before time  $\mu_1^\eta$  one has  $X_1 < Y_1$ . When  $X_1(t) \leq Z_1(t) \leq Y_1(t)$  for every  $t$ , a non-randomized  $\varepsilon$ -optimal stopping time exists (Lepeltier and Maingueneau, 1984). Laraki and Solan (2002, Section 4.1) use this observation to conclude the following.

**Lemma 15** *If  $Z_i(t) \in \text{co}\{X_i(t), Y_i(t)\}$  for every  $t \geq 0$  and each  $i = 1, 2$ , then the simple randomized stopping time  $\varphi_i^\eta$  in Lemma 14 can be taken to be non-randomized (that is, the delay of both players is 0).*

## 4 The Non-Zero-Sum Case

In the present section we prove Theorems 5 and 6. Fix  $\varepsilon > 0$  once and for all.

Let  $\delta_0$  (resp.  $\delta_1, \delta_2$ ) be a positive  $\mathcal{F}_\tau$ -measurable function that satisfies the following inequalities for each  $i \in \{1, 2\}$  and for the stopping time  $\tau = 0$  (resp.  $\tau = \mu_1^\eta, \tau = \mu_2^\eta$ ). Such  $\delta_0$  (resp.  $\delta_1, \delta_2$ ) exists because the processes  $(X_i, Y_i, v_i)_{i=1,2}$  are right continuous.

$$\mathbf{P}(\sup_{\rho \in [0, \delta_0]} |X_i(\tau) - X_i(\tau + \rho)| > \varepsilon) \leq \varepsilon, \quad (19)$$

$$\mathbf{P}(\sup_{\rho \in [0, \delta_0]} |Y_i(\tau) - Y_i(\tau + \rho)| > \varepsilon) \leq \varepsilon, \quad (20)$$

$$\mathbf{P}(\sup_{\rho \in [0, \delta_0]} |v_i(\tau) - v_i(\tau + \rho)| > \varepsilon) \leq \varepsilon. \quad (21)$$

We divide the set  $\Omega$  into six  $\mathcal{F}_0$ -measurable subsets. For each of these subsets we then define a pair of randomized stopping times  $(\varphi_1^*, \varphi_2^*)$ , and we prove that, when restricted to each set, this pair is a  $k\varepsilon$ -equilibrium, for some  $0 \leq k \leq 13$ . It will then follow that  $(\varphi_1^*, \varphi_2^*)$ , when viewed as a randomized stopping time on  $\Omega$ , is a  $78\varepsilon$ -equilibrium. The partition is similar to that in Laraki, Solan and Vieille (2005), and only the treatment on the last subset is different.

Denote by  $\psi_i(t, \varepsilon)$  an  $\varepsilon$ -optimal stopping time of player  $i$  in the game  $\Gamma_{3-i}(t)$ ; thus, the randomized stopping time  $\psi_i(t, \varepsilon)$  is a punishment strategy against player  $3-i$ , as it ensures that his payoff will not exceed  $v_{3-i}(t) + \varepsilon$ .

**Part 1:** The set  $A_1 := \{X_1(0) \geq v_1(0)\} \cap \{X_2(0) \geq Z_2(0)\}$ .

We prove that when restricted to the set  $A_1$ , the pair  $(\varphi_1^*, \varphi_2^*)$  that is defined as follows is a  $4\varepsilon$ -equilibrium:

- $\varphi_1^* = 0$ : player 1 stops at time 0.
- $\varphi_2^* = \psi_2(\delta_0, \varepsilon)$ : If player 1 does not stop before time  $\delta_0$ , player 2 punishes him in the game  $\Gamma_1(\delta_0)$  that starts at time  $\delta_0$ .

If no player deviates, the game is stopped by player 1, and the payoff is

$$\gamma(\varphi_1^*, \varphi_2^* \mid \mathcal{F}_0) = (X_1(0), X_2(0)) \text{ on } A_1.$$

We argue that player 2 cannot profit by deviating. Indeed, let  $\lambda_2$  be any non-randomized stopping time of player 2. Then on  $A_1$

$$\gamma_2(\varphi_1^*, \lambda_2 \mid \mathcal{F}_0) = Z_2(0)\mathbf{1}_{A_1 \cap \{\lambda_2=0\}} + X_2(0)\mathbf{1}_{A_1 \cap \{\lambda_2>0\}} \leq X_2(0) = \gamma_2(\varphi_1^*, \varphi_2^* \mid \mathcal{F}_0),$$

and the claim follows.

We now argue that on  $A_1$  player 1 cannot profit more than  $4\varepsilon$  by deviating from  $\varphi_1^*$ . Let  $\lambda_1$  be any non-randomized stopping time of player 1. Then by the definition of  $\varphi_2^*$ , on  $A_1$

$$\gamma_1(\lambda_1, \varphi_2^* \mid \mathcal{F}_0) \leq \mathbf{E}[X_1(\lambda_1)\mathbf{1}_{\{\lambda_1<\delta_0\}} + (v_1(\delta_0) + \varepsilon)\mathbf{1}_{\{\delta_0 \leq \lambda_1\}} \mid \mathcal{F}_0].$$

By (19), (21), and since  $X_1(0) \geq v_1(0)$  on  $A_1$ , it follows that on  $A_1$

$$\mathbf{P}(\gamma_1(\lambda_1, \varphi_2^* \mid \mathcal{F}_0) > \mathbf{E}[X_1(0)\mathbf{1}_{\{\lambda_1<\delta_0\}} + (X_1(0) + \varepsilon)\mathbf{1}_{\{\delta_0 \leq \lambda_1\}} \mid \mathcal{F}_0] + \varepsilon) \leq 2\varepsilon.$$

Since  $\gamma_1(\varphi_1^*, \varphi_2^* \mid \mathcal{F}_0) = X_1(0)$  on  $A_1$  it follows that

$$\mathbf{P}(A_1 \cap \{\gamma_1(\lambda_1, \varphi_2^* \mid \mathcal{F}_0) > \gamma_1(\varphi_1^*, \varphi_2^* \mid \mathcal{F}_0) + 2\varepsilon\}) \leq 2\varepsilon, \quad (22)$$

and the desired results follows.

**Part 2:** The set  $A_2 := \{Z_2(0) > X_2(0)\} \cap \{Z_1(0) \geq Y_1(0)\}$ .

We prove that when restricted to the set  $A_2$ , the pair  $(\varphi_1^*, \varphi_2^*)$  that is defined as follows is a 0-equilibrium:

- $\varphi_1^* = 0$ : player 1 stops at time 0.
- $\varphi_2^* = 0$ : player 2 stops at time 0.

If no player deviates, both players stop at time 0, and the payoff is

$$\gamma(\varphi_1^*, \varphi_2^* \mid \mathcal{F}_0) = (Z_1(0), Z_2(0)) \text{ on } A_2.$$

To see that player 1 cannot profit by deviating, fix an arbitrary non-randomized stopping time  $\lambda_1$  for player 1. On  $A_2$  one has

$$\gamma_1(\lambda_1, \varphi_2^* \mid \mathcal{F}_0) = Z_1(0)\mathbf{1}_{\{\lambda_1=0\}} + Y_1(0)\mathbf{1}_{\{\lambda_1>0\}} \leq Z_1(0) = \gamma_1(\varphi_1^*, \varphi_2^* \mid \mathcal{F}_0), \quad (23)$$

as desired. A symmetric argument shows that player 2 cannot profit by deviating either.

**Part 3:** The set  $A_3 := \{Y_1(0) > Z_1(0)\} \cap \{Y_2(0) \geq v_2(0)\}$ .

The case of the set  $A_3$  is analogous to Part 1: when restricted to  $A_3$ , the pair of randomized stopping times in which player 2 stops at time 0, and player 1 plays an  $\varepsilon$ -optimal stopping time  $\psi_1(\delta_0, \varepsilon)$  in the game  $\Gamma_2(\delta_0)$ , is a  $4\varepsilon$ -equilibrium.

**Part 4:** The set  $A_4 := \{X_1(0) \geq v_1(0)\} \cap \{X_2(0) > Y_2(0)\}$ .

We prove that when restricted to the set  $A_4$ , the pair  $(\varphi_1^*, \varphi_2^*)$  that is defined as follows is a  $6\varepsilon$ -equilibrium:

- $\varphi_1^*(r, \cdot) = r\delta_0$ : player 1 stops at a random time between time 0 and time  $\delta_0$ .
- $\varphi_2^* = \psi_2(\delta_0, \varepsilon)$ : If player 1 does not stop before time  $\delta_0$ , player 2 punishes him in the game  $\Gamma_1(\delta_0)$  that starts at time  $\delta_0$ .

If no player deviates, the game is stopped by player 1 before time  $\delta_0$ , and by (19) the payoff is within  $2\varepsilon$  of  $(X_1(0), X_2(0))$ :

$$\mathbf{P}(A_4 \cap \{|\gamma_i(\varphi_1^*, \varphi_2^*) - X_i(0)| > \varepsilon\}) \leq \varepsilon. \quad (24)$$

The same argument<sup>3</sup> as in Part 1 shows that

$$\mathbf{P}(A_4 \cap \{\gamma_1(\lambda_1, \varphi_2^* \mid \mathcal{F}_0) > \gamma_1(\varphi_1^*, \varphi_2^* \mid \mathcal{F}_0) + 3\varepsilon\}) \leq 3\varepsilon. \quad (25)$$

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<sup>3</sup>The additional  $\varepsilon$  arises because in Part 1 we had  $\gamma_1(\varphi_1^*, \varphi_2^*) = X_1(0)$ , whereas in Part 4 we have  $\mathbf{P}(A_4 \cap \{\gamma_1(\varphi_1^*, \varphi_2^*) < X_1(0) - \varepsilon\}) \leq \varepsilon$ .

It follows that player 1 cannot profit more than  $6\varepsilon$  by deviating from  $\varphi_1^*$ .

We now argue that player 2 cannot profit more than  $5\varepsilon$  by deviating from  $\varphi_2^*$ . Fix a non-randomized stopping time  $\lambda_2$  for player 2. On  $A_4$  we have  $\varphi_1^* \leq \delta_0$ , and  $\mathbf{P}(A_4 \cap \{\varphi_1^* = \lambda_2\}) = 0$ , and therefore

$$\gamma_2(\varphi_1^*, \lambda_2) = \mathbf{E}[X_2(\varphi_1^*)\mathbf{1}_{\{\varphi_1^* < \lambda_2\}} + Y_2(\lambda_2)\mathbf{1}_{\{\lambda_2 < \varphi_1^*\}} \mid \mathcal{F}_0] \text{ on } A_4.$$

By (19) and (20),

$$\mathbf{P}(\gamma_2(\varphi_1^*, \lambda_2) > \mathbf{E}[(X_2(0) + \varepsilon)\mathbf{1}_{\{\varphi_1^* < \lambda_2\}} + (Y_2(0) + \varepsilon)\mathbf{1}_{\{\lambda_2 < \varphi_1^*\}} \mid \mathcal{F}_0]) \leq 2\varepsilon.$$

Because  $X_2(0) > Y_2(0)$  on  $A_4$  we have

$$\mathbf{P}(\gamma_2(\varphi_1^*, \lambda_2) > X_2(0) + \varepsilon) \leq 2\varepsilon.$$

Together with (24) we deduce that

$$\mathbf{P}(\gamma_2(\varphi_1^*, \lambda_2) > \gamma_2(\varphi_1^*, \varphi_2^*) + 2\varepsilon) \leq 3\varepsilon,$$

and the claim follows.

**Part 5:** The set  $A_5 := \{X_1(0) \geq v_1(0)\} \setminus (A_1 \cup A_2 \cup A_3 \cup A_4)$ .

We claim that  $\mathbf{P}(A_5) = 0$ . Since  $X_1(0) \geq v_1(0)$  on  $A_5$ , and since  $A_5 \cap A_1 = \emptyset$ , it follows that  $X_2(0) < Z_2(0)$  on  $A_5$ . Since  $A_5 \cap A_2 = \emptyset$ , it follows that  $Z_1(0) < Y_1(0)$  on  $A_5$ . Since  $A_5 \cap A_3 = \emptyset$ , it follows that  $Y_2(0) < v_2(0)$  on  $A_5$ . Since  $A_5 \cap A_4 = \emptyset$ , it follows that  $Y_2(0) \geq X_2(0)$  on  $A_5$ . Lemma 9 then implies that

$$Y_2(0) < v_2(0) \leq \max\{X_2(0), Y_2(0)\} = Y_2(0) \text{ on } A_5,$$

which in turn implies that  $\mathbf{P}(A_5) = 0$ , as claimed.

The union  $A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5$  includes the set  $\{X_1(0) \geq v_1(0)\}$ . Thus, when restricted to this set, the game has a  $7\varepsilon$ -equilibrium. By symmetric arguments, a  $6\varepsilon$ -equilibrium exists on the set  $\{Y_2(0) \geq v_2(0)\}$ . We now construct a  $13\varepsilon$ -equilibrium on the remaining set,  $\{X_1(0) < v_1(0)\} \cap \{Y_2(0) < v_2(0)\}$ .

**Part 6:** The set  $A_6 := \{X_1(0) < v_1(0)\} \cap \{Y_2(0) < v_2(0)\}$ .

Fix  $\eta > 0$ , and for each  $i \in \{1, 2\}$  let  $\varphi_i^\eta$  be a simple randomized stopping time with basis  $\mu_i^\eta$  and delay at most  $\delta_i$  that satisfies Eq. (18) for every stopping time  $\lambda_{3-i} \geq \mu_i^\eta$  (see Lemma 14). Let  $\psi_1(\mu_2^\eta + \delta_2, \varepsilon)$  (resp.  $\psi_2(\mu_1^\eta + \delta_1, \varepsilon)$ ) be a simple randomized  $\varepsilon$ -optimal stopping time for player 1 in the game  $\Gamma_2(\mu_2^\eta + \delta_2)$  (resp. in the game  $\Gamma_1(\mu_1^\eta + \delta_1)$ ); that is, a stopping time that achieves the infimum in (8) up to  $\varepsilon$ , for  $t = \mu_2^\eta + \delta_2$  (resp. the infimum in (7) up to  $\varepsilon$ , for  $t = \mu_1^\eta + \delta_1$ ).

Set  $\mu^\eta = \min\{\mu_1^\eta, \mu_2^\eta\}$ . We further divide  $A_6$  into six  $\mathcal{F}_{\mu^\eta}$ -measurable subsets; the definition of  $(\varphi_1^*, \varphi_2^*)$  is different in each subset, and is given in the second and third columns of Table 1. Under  $(\varphi_1^*, \varphi_2^*)$  the game will be stopped at time  $\mu^\eta$  or during a short interval after time  $\mu^\eta$ , if  $\mu^\eta < \infty$ , and will not be stopped if  $\mu^\eta = \infty$ .

Subset	$\varphi_1^*$	$\varphi_2^*$	$\gamma_1(\varphi_1^*, \varphi_2^*)$
$A_{61} := A_6 \cap \{\mu_1^\eta < \mu_2^\eta\}$	$\varphi_1^\eta$	$\psi_2(\mu^\eta + \delta_1)$	$\geq X_1(\mu^\eta) - 2\varepsilon$
$A_{62} := A_6 \cap \{\mu_2^\eta < \mu_1^\eta\}$	$\psi_1(\mu^\eta + \delta_2)$	$\varphi_2^\eta$	$\geq Y_1(\mu^\eta) - 2\varepsilon$
$A_{63} := A_6 \cap \{\mu_1^\eta = \mu_2^\eta = \infty\}$	$\infty$	$\infty$	$= \xi_1$
$A_{64} := A_6 \cap \{\mu_1^\eta = \mu_2^\eta < \infty\} \cap \{Z_1(\mu_1^\eta) < Y_1(\mu_1^\eta)\}$	$\psi_1(\mu_1^\eta + \delta_2, \varepsilon)$	$\mu^\eta$	$= Y_1(\mu^\eta)$
$A_{65} := A_6 \cap \{\mu_1^\eta = \mu_2^\eta < \infty\} \cap \{Z_2(\mu_1^\eta) < X_2(\mu_1^\eta)\}$	$\mu^\eta$	$\psi_2(\mu^\eta + \delta_1, \varepsilon)$	$= X_1(\mu^\eta)$
$A_{66} := A_6 \cap \{\mu_1^\eta = \mu_2^\eta < \infty\} \cap \{Y_1(\mu_1^\eta) \leq Z_1(\mu_1^\eta)\} \cap \{X_2(\mu^\eta) \leq Z_2(\mu^\eta)\}$	$\mu^\eta$	$\mu^\eta$	$= Z_1(\mu^\eta)$

Table 1: The randomized stopping times  $(\varphi_1^*, \varphi_2^*)$  on  $A_6$ , with the payoff to player 1.

We argue that when restricted to  $A_6$ , the pair  $(\varphi_1^*, \varphi_2^*)$  is a  $13\varepsilon$ -equilibrium. Note that the roles of the two players in the definition of  $(\varphi_1^*, \varphi_2^*)$  are symmetric:  $\varphi_1^* = \varphi_2^*$  on  $A_{63}$  and  $A_{66}$ , and the role of player 1 (resp. player 2) in  $A_{61}$  and  $A_{64}$  is similar to the role of player 2 (resp. player 1) in  $A_{62}$  and  $A_{65}$ . To prove that  $(\varphi_1^*, \varphi_2^*)$  is a  $13\varepsilon$ -equilibrium it is therefore sufficient to prove that the probability that player 1 can profit more than  $3\varepsilon$  by deviating from  $\varphi_1^*$  is at most  $10\varepsilon$ .

We start by bounding the payoff  $\gamma_1(\varphi_1^*, \varphi_2^* | \mathcal{F}_{\mu^\eta})$  (the bound that we derive appears on



the right-most column in Table 1), and by showing that

$$\gamma_1(\varphi_1^*, \varphi_2^* \mid \mathcal{F}_{\mu^\eta}) \geq v_1(\mu^\eta) - 3\varepsilon - \eta \text{ on } A_6 \setminus A_{63}. \quad (26)$$

We prove this in turn on each of the sets  $A_{61}, \dots, A_{66}$ :

- On  $A_{61}$  we have  $\mu^\eta = \mu_1^\eta$ , and the game is stopped by player 1 between times  $\mu^\eta$  and  $\mu^\eta + \delta_1$ , so that by (19) we have

$$\mathbf{P}(A_{61} \cap \{\gamma_1(\varphi_1^*, \varphi_2^* \mid \mathcal{F}_{\mu^\eta}) < X_1(\mu^\eta) - \varepsilon\}) \leq \varepsilon. \quad (27)$$

By (13) we have  $X_1(\mu^\eta) \geq v_1(\mu^\eta) - \eta$ , and therefore (26) holds on  $A_{61}$ .

- On  $A_{62}$  we have  $\mu^\eta = \mu_2^\eta$ , and the game is stopped by player 2 between times  $\mu^\eta$  and  $\mu^\eta + \delta_2$ , so that by (20) we have

$$\mathbf{P}(A_{62} \cap \{\gamma_1(\varphi_1^*, \varphi_2^* \mid \mathcal{F}_{\mu^\eta}) < Y_1(\mu^\eta) - \varepsilon\}) \leq \varepsilon. \quad (28)$$

By (15) we have  $X_1(\mu^\eta) < Y_1(\mu^\eta)$  on  $A_{62}$ , so that by Lemma 9 we have  $Y_1(\mu^\eta) \geq v_1(\mu^\eta)$ . It follows that (26) holds on  $A_{62}$ .

- On  $A_{63}$  no player ever stops, and therefore  $\gamma_1(\varphi_1^*, \varphi_2^* \mid \mathcal{F}_{\mu^\eta}) = \xi_1$ .
- On  $A_{64}$  player 2 stops at time  $\mu^\eta$ , and therefore  $\gamma_1(\varphi_1^*, \varphi_2^* \mid \mathcal{F}_{\mu^\eta}) = Y_1(\mu^\eta)$ . By Lemma 10, on  $A_{64}$  we have

$$v_1(\mu^\eta) \leq \max\{Y_1(\mu^\eta), Z_1(\mu^\eta)\} = Y_1(\mu^\eta),$$

and therefore (26) holds on  $A_{64}$ .

- On  $A_{65}$  player 1 stops at time  $\mu^\eta$ , and therefore  $\gamma_1(\varphi_1^*, \varphi_2^* \mid \mathcal{F}_{\mu^\eta}) = X_1(\mu^\eta)$ . By (13) we have  $X_1(\mu^\eta) \geq v_1(\mu^\eta) - \eta$ , and therefore (26) holds on  $A_{65}$ .
- On  $A_{66}$  both players stop at time  $\mu^\eta$ , and therefore  $\gamma_1(\varphi_1^*, \varphi_2^* \mid \mathcal{F}_{\mu^\eta}) = Z_1(\mu^\eta)$ . By Lemma 10 on this set we have

$$v_1(\mu^\eta) \leq \max\{Y_1(\mu^\eta), Z_1(\mu^\eta)\} = Z_1(\mu^\eta),$$

and therefore (26) holds on  $A_{66}$ .

Fix a stopping time  $\lambda_1$  for player 1. To complete the proof of Theorem 5 we prove that

$$\mathbf{P}(A_6 \cap \{\gamma_1(\lambda_1, \varphi_2^*) > \gamma_1(\varphi_1^*, \varphi_2^*) + 3\varepsilon\}) \leq 10\varepsilon.$$

- On the set  $A_6 \cap \{\lambda_1 < \mu^\eta\}$  we have by the definition of  $\mu_1^\eta$ , since  $\mu^\eta \leq \mu_1^\eta$ , by Lemma 13, and by (26),

$$\begin{aligned} \gamma_1(\lambda_1, \varphi_2^* \mid \mathcal{F}_{\lambda_1}) &= X_1(\lambda_1) \\ &< v_1(\lambda_1) - \eta \\ &\leq \mathbf{E}[v_1(\mu^\eta) \mathbf{1}_{A_6 \cap \{\lambda_1 < \mu^\eta < \infty\}} + \xi_1 \mathbf{1}_{A_6 \cap \{\lambda_1 < \mu^\eta = \infty\}} \mid \mathcal{F}_{\lambda_1}] - \eta \\ &\leq \gamma_1(\varphi_1^*, \varphi_2^* \mid \mathcal{F}_{\lambda_1}) + 3\varepsilon, \end{aligned} \tag{29}$$

where the last inequality holds by (26) and because the payoff of player 1 on  $A_{63}$  is  $\xi_1$ .

- On the set  $A_{61} \cap \{\mu^\eta \leq \lambda_1\}$  we have by the definition of  $\varphi_2^*$

$$\gamma_1(\lambda_1, \varphi_2^* \mid \mathcal{F}_{\mu^\eta}) = \mathbf{E}[X_1(\lambda_1) \mathbf{1}_{\{\lambda_1 \leq \mu^\eta + \delta_1\}} + (v_1(\mu^\eta + \delta_1) + \varepsilon) \mathbf{1}_{\{\mu^\eta + \delta_1 < \lambda_1\}} \mid \mathcal{F}_{\mu^\eta}].$$

By (19), (21) and (13) we have

$$\mathbf{P}(\gamma_1(\lambda_1, \varphi_2^* \mid \mathcal{F}_{\mu^\eta}) > X_1(\mu^\eta) + 2\varepsilon) \leq 2\varepsilon.$$

By (27) we deduce that

$$\mathbf{P}(\gamma_1(\lambda_1, \varphi_2^* \mid \mathcal{F}_{\mu^\eta}) > \gamma_1(\varphi_1^*, \varphi_2^* \mid \mathcal{F}_{\mu^\eta}) + 3\varepsilon) \leq 3\varepsilon. \tag{30}$$

- On the set  $A_{62} \cap \{\mu^\eta \leq \lambda_1\}$  we have by the definition of  $\varphi_2^*$

$$\gamma_1(\lambda_1, \varphi_2^* \mid \mathcal{F}_{\mu^\eta}) = \mathbf{E}[X_1(\lambda_1) \mathbf{1}_{\{\mu^\eta \leq \lambda_1 < \varphi_2^*\}} + Y_1(\varphi_2^*) \mathbf{1}_{\{\varphi_2^* \leq \lambda_1\}} \mid \mathcal{F}_{\mu^\eta}].$$

By (19), (20), since  $\mu_2^\eta < \mu_1^\eta$  on  $A_{62}$ , and by (15),

$$\mathbf{P}(\gamma_1(\lambda_1, \varphi_2^* \mid \mathcal{F}_{\mu^\eta}) > \mathbf{E}[(Y_1(\mu^\eta) + \varepsilon) \mathbf{1}_{\{\mu^\eta \leq \lambda_1 < \varphi_2^*\}} + (Y_1(\mu^\eta) + \varepsilon) \mathbf{1}_{\{\varphi_2^* \leq \lambda_1\}} \mid \mathcal{F}_{\mu^\eta}]) \leq 2\varepsilon.$$

By (28) we deduce that

$$\mathbf{P}(A_{62} \cap \{\gamma_1(\lambda_1, \varphi_2^* \mid \mathcal{F}_{\mu^\eta}) > \gamma_1(\varphi_1^*, \varphi_2^* \mid \mathcal{F}_{\mu^\eta}) + 2\varepsilon\}) \leq 3\varepsilon. \tag{31}$$

- On the set  $A_{63} \cap \{\mu_2^\eta \leq \lambda_1\}$  we have  $\mu^\eta = \lambda_1 = \infty$ , so that

$$\gamma_1(\varphi_1^*, \varphi_2^* \mid \mathcal{F}_{\mu^\eta}) = \xi_1 = \gamma_1(\lambda_1, \varphi_2^* \mid \mathcal{F}_{\mu^\eta}) \text{ on } A_{63} \cap \{\mu_2^\eta \leq \lambda_1\}. \quad (32)$$

- On the set  $A_{64} \cap \{\mu^\eta \leq \lambda_1\}$  we have

$$\begin{aligned} \gamma_1(\lambda_1, \varphi_2^* \mid \mathcal{F}_{\mu^\eta}) &= \mathbf{E}[Z_1(\mu^\eta) \mathbf{1}_{\{\lambda_1 = \mu^\eta\}} + Y_1(\mu^\eta) \mathbf{1}_{\{\mu^\eta < \lambda_1\}}] \\ &\leq Y_1(\mu^\eta) = \gamma_1(\varphi_1^*, \varphi_2^* \mid \mathcal{F}_{\mu^\eta}). \end{aligned} \quad (33)$$

- On the set  $A_{65} \cap \{\mu^\eta \leq \lambda_1\}$  we have by the definition of  $\varphi_2^*$

$$\gamma_1(\lambda_1, \varphi_2^* \mid \mathcal{F}_{\mu^\eta}) = \mathbf{E}[X_1(\lambda_1) \mathbf{1}_{\{\mu^\eta \leq \lambda_1 < \mu^\eta + \delta_1\}} + (v_1(\mu^\eta + \delta_1) + \varepsilon) \mathbf{1}_{\{\mu^\eta + \delta_1 \leq \lambda_1\}} \mid \mathcal{F}_{\mu^\eta}].$$

By (19), (21) and (13) we have

$$\mathbf{P}(A_{65} \cap \{\mu^\eta \leq \lambda_1\} \cap \{\gamma_1(\lambda_1, \varphi_2^* \mid \mathcal{F}_{\mu^\eta}) > X_1(\mu^\eta) + 2\varepsilon\}) \leq 2\varepsilon.$$

Because  $\gamma_1(\varphi_1^*, \varphi_2^* \mid \mathcal{F}_{\mu^\eta}) = X_1(\mu^\eta)$  on  $A_{65}$ , we obtain

$$\mathbf{P}(A_{65} \cap \{\mu^\eta \leq \lambda_1\} \cap \{\gamma_1(\lambda_1, \varphi_2^* \mid \mathcal{F}_{\mu^\eta}) > \gamma_1(\varphi_1^*, \varphi_2^* \mid \mathcal{F}_{\mu^\eta}) + 2\varepsilon\}) \leq 2\varepsilon. \quad (34)$$

- On the set  $A_{66} \cap \{\mu^\eta \leq \lambda_1\}$  we have

$$\begin{aligned} \gamma_1(\lambda_1, \varphi_2^* \mid \mathcal{F}_{\mu^\eta}) &= Z_1(\mu^\eta) \mathbf{1}_{\{\lambda_1 = \mu^\eta\}} + Y_1(\mu^\eta) \mathbf{1}_{\{\mu^\eta < \lambda_1\}} \\ &\leq Z_1(\mu^\eta) = \gamma_1(\varphi_1^*, \varphi_2^* \mid \mathcal{F}_{\mu^\eta}). \end{aligned} \quad (35)$$

From (30), (31), (32), (33), (34) and (35) we deduce that on  $A_6 \cap \{\mu^\eta \leq \lambda_1\}$

$$\mathbf{P}(\gamma_1(\lambda_1, \varphi_2^* \mid \mu^\eta) > \gamma_1(\varphi_1^*, \varphi_2^* \mid \mu^\eta) + 3\varepsilon) \leq 10\varepsilon. \quad (36)$$

Because (29) and (36) hold for every stopping time  $\lambda_1$  for player 1, it follows that  $(\varphi_1^*, \varphi_2^*)$  is a  $13\varepsilon$ -equilibrium on  $A_6$ , as desired.

**Proof of Theorem 6.** To prove that if  $Z_1(t) \in co\{X_1(t), Y_1(t)\}$  and  $Z_2(t) \in co\{X_2(t), Y_2(t)\}$  for every  $t \geq 0$ , then there is a pair of non-randomized stopping times that form an  $\varepsilon$ -equilibrium, we are going to check where randomized stopping times were used in the proof of Theorem 5, and we will see how in each case one can use non-randomized stopping times instead of randomized stopping times.

1. In Part 1 (and in the analogue part 3) we used a punishment strategy  $\psi_1(\delta_0, \varepsilon)$  that in general is a non-randomized stopping time. However, by Lemma 15, when  $Z_2(t) \in \text{co}\{X_2(t), Y_2(t)\}$  for every  $t \geq 0$ , this randomized stopping time can be taken to be non-randomized.
2. In Part 4 we used, in addition to the punishment strategy  $\psi_2(\delta_0, \varepsilon)$ , a simple randomized stopping time for player 1. The set that we were concerned with in part 4 was the set  $A_4 := \{X_1(0) \geq v_1(0)\} \cap \{X_2(0) > Y_2(0)\}$ . Because  $Z_2(0) \in \text{co}\{X_2(0), Y_2(0)\}$

$$X_2(0) \geq Z_2(0) \geq Y_2(0).$$

But then the following pair of non-randomized stoping times is a  $3\varepsilon$ -equilibrium when restricted to  $A_4$ :

- $\varphi_1^* := 0$ : player 1 stops at time 0.
- $\varphi_2^* := \psi_2(\delta_0, \varepsilon)$ : if player 1 does not stop before time  $\delta_0$ , player 2 punishes him (with a non-randomized stopping time; see first item) in the game  $\Gamma_1(\delta_0)$ .

3. In Part 6 randomization was used both for punishment (on  $A_{61}$ ,  $A_{62}$ ,  $A_{64}$  and  $A_{65}$ ) and for stopping (on  $A_{61}$  and  $A_{62}$ ). As mentioned above, under the assumptions of Theorem 6, for punishment one can use non-randomized stopping times. We now argue that one can modify the definition of  $(\varphi_1^*, \varphi_2^*)$  on  $A_{61}$  and  $A_{62}$  so as to obtain a non-randomized equilibrium. Because of the symmetry between  $A_{61}$  and  $A_{62}$ , we show how to modify the construction only on  $A_{61}$ .

On  $A_{61}$  we have  $\mu_1^\eta < \mu_2^\eta$ , so that by (15) we have  $Y_2(\mu_1^\eta) < X_2(\mu_2^\eta)$ . Because  $Z_2(\mu_1^\eta) \in \text{co}\{X_2(\mu_1^\eta), Y_2(\mu_1^\eta)\}$  it follows that  $Y_2(\mu_1^\eta) \leq Z_2(\mu_1^\eta) \leq X_2(\mu_2^\eta)$ . But then the following pair of non-randomized stoping times is a  $3\varepsilon$ -equilibrium on  $A_{61}$ :

- $\varphi_1^* := \mu_1^\eta$ : player 1 stops at time  $\mu_1^\eta$ .
- $\varphi_2^* := \psi_2(\mu_1^\eta + \delta_1, \varepsilon)$ : if player 1 does not stop before time  $\mu_1^\eta + \delta_1$ , player 2 punishes him (with a non-randomized stopping time; see first item) in the game  $\Gamma_1(\mu_1^\eta + \delta_1)$ .

■

## A The result of Laraki and Solan (2005)

As mentioned before, Laraki and Solan (2005) proved Theorem 5 for two-player zero-sum Dynkin games. We need the stronger version that is stated in Lemma 7, where the payoff is conditioned on the  $\sigma$ -algebra  $\mathcal{F}_t$ . It turns out that the arguments used by Laraki and Solan (2005) prove this case as well, when one uses the following Lemma instead of Lemma 4 in Laraki and Solan (2005).

**Lemma 16** *Let  $X$  be a right-continuous process. For every stopping time  $\lambda$  and every positive  $\mathcal{F}_\lambda$ -measurable function  $\varepsilon$  there is a positive  $\mathcal{F}_\lambda$ -measurable and bounded function  $\delta$  such that:*

$$|X(\lambda) - \mathbf{E}[X(\rho) \mid \mathcal{F}_\lambda]| \leq \varepsilon, \quad (37)$$

*for every stopping time  $\rho$  that satisfies  $\lambda \leq \rho \leq \lambda + \delta$ .*

**Proof.** Because the process  $X$  is right continuous, the function  $w \mapsto \mathbf{E}[X(\lambda + w) \mid \mathcal{F}_\lambda]$  is right-continuous at  $w = 0$  on  $\Omega$ , and it is equal to  $X(\lambda)$  at  $w = 0$ . By defining

$$\delta' = \frac{1}{2} \sup\{w > 0: |X(\lambda) - \mathbf{E}[X(\lambda + w) \mid \mathcal{F}_\lambda]| \leq \varepsilon\},$$

we obtain a positive  $\mathcal{F}_\lambda$ -measurable function such that (37) is satisfied for every stopping time  $\rho$ ,  $\lambda \leq \rho \leq \lambda + \delta'$ . The proof of the Lemma is complete by setting  $\delta = \min\{\delta', 1\}$ . ■

This Lemma can also be used to adapt the proof of Proposition 7 in Laraki and Solan (2005) in order to prove Lemma 8, which states that the value process is right continuous.

One can use Lemma 16 to improve some of the bounds given in Section 4. We chose not to use this Lemma in the paper, so as to unify the arguments given for the various bounds.

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